is more pronounced: The velocity profiles in the flows are not the same in all the sections, and the ratio of the flow rates K in the majority of the sections is less than 0.7. The tabulated data also confirm that a reduction in Ra is accompanied by an increase in the three-dimensional nature of the flow.

Hence, the presence of transverse ribbing on one of the walls of a vertical convection layer of liquid causes the appearance of three-dimensional flow. The presence of this phenomenon in the flow eliminates the possibility of calculating the flow parameters using methods applicable to plane-parallel flows and requires a special study.

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DRAINING LIQUID-FILM SOLITONS

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One-dimensional solitary waves (solitons) which can move on the surface of a thin layer of a viscous liquid, draining in a vertical plane, are investigated. The first (experimental) description of such waves was given in [1]; later, quantitative measurements of their characteristics were carried out [2, 3] and attempts were made to explain them theoretically. In [4-6] the nature of these waves are discussed and some of their properties are pointed out. In [7] the view is put forward that a stationary solution is the limiting solution of a quasiharmonic type as the wave number is reduced. In this paper we carry out a qualitative analysis of the evolution of a nonstationary soliton and on the basis of the analysis we explain its shape. The fundamental characteristics of a stationary soliton (the amplitude and velocity) are calculated and the results obtained are compared with experimental data.

1. The equation for the waves in a vertically draining film of viscous liquid for low Reynolds numbers is well known and can be obtained by different methods. Assuming long weakly nonlinear waves, the equation takes the form

$$\varphi_t + 3\varphi_x + \varphi\varphi_x + \mathrm{Re}\,\varphi_{xx} + W\varphi_{xxxx} = 0, \tag{1.1}$$

where $\varphi = 6(h - \langle h\rangle)/\langle h\rangle$; h is the local thickness of the film, $\langle h\rangle$ is the thickness of the film averaged over the length, t is dimensionless time, x is the dimensionless vertical coordinate (downwards) (the scale for measuring the length is $\langle h\rangle$, the scale for measuring time is $3\nu g^{-1}\langle h\rangle^{-1}$, ν is the viscosity, and g is the acceleration due to gravity), $\mathrm{Re} = 2g\langle h\rangle^3/5\nu^2$ is Reynolds number, and $W = \sigma/\rho g\langle h\rangle^2$ is Weber's number (σ is the surface tension and ρ is the density of the liquid).

Together with the Burgers and Korteweg–de Vries equations, Eq. (1.1) belongs to the number of so-called nonlinear evolution equations. The treatment of the quantities φ and $\varphi^2/2$ as the momentum density and energy density is common to these equations (this treatment is related to the Galilean invariance of the nonlinear evolution equations). By confining ourselves henceforth to considering only solitary waves (solitons) for which $\varphi \to 0$ as $x \to \pm\infty$, in which case there are integrals of the form $\int_{-\infty}^{\infty} \varphi^2(x, t)\,dx$, $\int_{-\infty}^{\infty} \varphi_x^2(x, t)\,dx$,

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$\int_{-\infty}^{\infty} \varphi_{xx}^2(x, t)\,dx$, we obtain from (1.1) the laws of variation of the momentum and the energy $\frac{\partial}{\partial t}\int_{-\infty}^{\infty} \varphi dx = 0$, $\frac{\partial}{\partial t}\int_{-\infty}^{\infty} \frac{\varphi^2}{2}\,dx = \mathrm{Re}\int_{-\infty}^{\infty} \varphi^2 dx - \mathrm{W}\int_{-\infty}^{\infty} \varphi_{xx}^2 dx$. It can be seen that the term Re φ_{xx} in Eq. (1.1) describes the pumping of energy into the wave proportional to the Reynolds number, while the term $W\varphi_{xxxx}$ describes the energy dissipation, where energy pumping occurs at the low frequencies and dissipation at the high frequencies.

We will now use the momentum of a soliton to define the idea of velocity: We will define the velocity of a soliton as the velocity of propagation of momentum (for a soliton, which is formed localized space and has a considerable spectral spread of wave numbers and in which energy exchange with the medium occurs by pumping and dissipation, the definition of the velocity as the group velocity at which the energy is propagated is not successful). By defining the position of the momentum as $\bar{x} = \int_{-\infty}^{\infty} x\varphi dx \Big/ \int_{-\infty}^{\infty} \varphi dx$, we obtain (using (1.1)) the following expression for the velocity of a soliton:

$$v_s = \frac{d\bar{x}}{dt} = \frac{\int_{-\infty}^{\infty} x\varphi_t dx}{\int_{-\infty}^{\infty} \varphi dx} = 3 + \frac{1}{2}\frac{\int_{-\infty}^{\infty} \varphi^2 dx}{\int_{-\infty}^{\infty} \varphi dx}.$$

Note that the definition of the velocity of a wave packet as the velocity of propagation of the momentum for linearized spectrally narrow wave packets leads to the usual group velocity. In fact, we will consider a perturbation having the form of a spectrally narrow wave packet with a carrier wave number k_0: $\varphi(x, t) = A(x, t)\exp(-ik_0 x + i\omega_0 t)$, where $A(x, t)$ is a slowly varying amplitude (decreasing as $x \to \pm\infty$). Expanding $A(x, t)$ in a Fourier integral we obtain

$$\varphi(x, t) = \int_{-\infty}^{\infty} \alpha(k)\, e^{i(k-k_0)x - i(\omega(k)-\omega_0)t}\,dk.$$

We further have

$$\bar{x} = \frac{\int_{-\infty}^{\infty} x\varphi dx}{\int_{-\infty}^{\infty} \varphi dx} = \frac{\int_{-\infty}^{\infty} \alpha(k)\, e^{-i(\omega-\omega_0)t}\,dk \int_{-\infty}^{\infty} x\, e^{i(k-k_0)x}\,dx}{\int_{-\infty}^{\infty} \alpha(k)\, e^{-i(\omega-\omega_0)t}\,dk \int_{-\infty}^{\infty} e^{i(k-k_0)x}\,dx} = \frac{-2\pi i\int_{-\infty}^{\infty} \alpha(k)\, e^{-i(\omega-\omega_0)t}\,\delta(k-k_0)\,dk}{2\pi\int_{-\infty}^{\infty} \alpha(k)\, e^{-i(\omega-\omega_0)t}\,\delta(k-k_0)\,dk} = i\frac{\alpha'(k_0)}{\alpha(k_0)} + t\left.\frac{d\omega}{dk}\right|_{k=k_0}.$$

From this we obtain $v_s = d\bar{x}/dt = d\omega/dk|_{k=k_0}$.

Turning now to a discussion of the properties of the solitons of Eq. (1.1) we note that in view of the conservation of the quantity $\int_{-\infty}^{\infty} \varphi dx$ we can introduce a time-invariant classification of solitons, namely, we will call a soliton positive (negative) if its momentum (which has the geometrical meaning of area) is positive (negative). In this case, it follows from the expression given above for v_s (the soliton velocity) that for a positive soliton $v_s > 3$, and for a negative soliton $v_s < 3$. We will recalculate (1.1) to the reference system $\xi = x - 3t$

$$\varphi_t + \varphi\varphi_\xi + \mathrm{Re}\,\varphi_{\xi\xi} + W\varphi_{\xi\xi\xi\xi} = 0. \tag{1.2}$$

It can be seen that (1.2) is invariant to the transformation $\xi \to -\xi$, $\varphi \to -\varphi$. Since this transformation converts a positive soliton into a negative one, we see that a negative soliton is simply an inversion (in the system of axes $\xi = x - 3t$ and $y - \langle h\rangle$) of a positive soliton. Hence, below, to be specific, we consider a positive soliton.

We will now make a qualitative analysis of the part played by the individual terms in Eq. (1.2) in the evolution of a (positive) soliton, regarding them as terms which make a contribution to φ_t. To do this we will consider the equations

$$\varphi_t + \varphi\varphi_\xi = 0,\ \varphi_t + \mathrm{Re}\,\varphi_{\xi\xi} = 0,\ \varphi_t + W\varphi_{\xi\xi\xi\xi} = 0. \tag{1.3}$$

The first of these equations describes a simple wave $\varphi = f(\xi - \varphi t)$. The motion of this wave is accompanied by an increase in the curvature of the leading front (in the ξ system) and a reduction in the curvature of the trailing front.

If $\varphi(t_0, \xi) \geq 0$, the leading front will be the right front (a simple positive wave moves to the right). This determines the role of the term $\varphi\varphi_\xi$.

The second of Eqs. (1.3) describes diffusion with a negative coefficient. Hence, the term $\mathrm{Re}\,\varphi_{\xi\xi}$ leads to a monotonic increase in the gradients in the wave (it is responsible for the instability of the trivial solution $\varphi \equiv 0$).

To explain the part played by the term $W\varphi_{\xi\xi\xi\xi}$ we consider the third of the equations (1.3). Its solution has the form $\varphi(\xi, t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \alpha(k)\exp(-ik\xi - Wk^4 t)dk$, $\alpha(k) = \int_{-\infty}^{\infty} \varphi(\xi, 0)\exp(ik\xi)\,d\xi$. Using the method of stationary phase, we obtain the asymptotic for large ξ (and for large t)

$$\varphi(\xi, t) = \sqrt{\frac{8}{3\pi}}(4Wt\xi^2)^{-1/6}\mathrm{Real}\left\{\alpha\left[\frac{\xi^{1/3}e^{-i\pi/6}}{(4Wt)^{1/3}}\right]\exp\left[-\frac{3}{8}\left(1 + i\sqrt{3}\right)\xi^{4/3}(4Wt)^{-1/3} + i\theta\right]\right\}. \tag{1.4}$$

It is seen from (1.4) that the term $W\varphi_{\xi\xi\xi\xi}$ leads to a reduction in the function φ and its gradients, but (and it is important to emphasize this) this reduction, unlike diffusion, has a nonmonotonic character, and has the form of decaying oscillations (on both sides). But it can also be seen that the phase of these oscillations is constant when $\xi^{4/3}t^{-1/3}$ = const, and, of course, the phase velocity of the oscillations is equal to $\xi \sim t^{-3/4}$. Since t is assumed to be large, the oscillations do not in practice move with respect to the soliton.

The combined action of all the terms in (1.2) is such that when a sufficiently coarse-amplitude (but mildly sloping) perturbation $\varphi(t_0, \xi) \geq 0$ acts, the term $\varphi\varphi_\xi$ initially plays a decisive role in its evolution, deforming this perturbation in such a way that the leading (in the ξ system) (right) front becomes steep, while the trailing (left) front becomes more mildly sloping. Since only the slope of the leading front becomes fairly large, at points of considerable slope considerable dispersion terms $\mathrm{Re}\,\varphi_{\xi\xi}$ and $W\varphi_{\xi\xi\xi\xi}$ arise, the actions of which are different: Whereas the first of these tends to increase the gradient φ, the second, on the other hand, decreases these gradients. In the final analysis it is the action of the term $W\varphi_{\xi\xi\xi\xi}$ which stabilizes the slope of the wavefront and prevents it from reversing.

Since, in addition, this term produces (stationary) oscillations attenuating along ξ, these oscillations also occupy part of the leading front (on the mildly sloping trailing front of the wave the smallness of the gradients enables one to neglect the effect of the dispersion terms so that, in particular, oscillations also do not occur there).

Since when $\varphi(t_0, \xi) \geq 0$ the velocity of propagation of a disturbance is positive in the $\xi = x - 3t$ system, it moves to the right both with respect to the laboratory reference system x, and with respect to the ξ system. Hence, in both reference systems the leading front is one and the same – the right front of the wave, and in connection with what was said above the soliton acquires with time the shape shown in Fig. 1a. When $\varphi(t_0, \xi) \leq 0$ the perturbation moves to the left with respect to the ξ reference system so that its steep front will be the left front. In the x reference system this front is the trailing front of the soliton. Since dispersion spreading occurs only at the steep front, in a negative soliton the trailing front will be spread (Fig. 1b).

This account agrees with the results of experimental observations [1–3]. However, experiments indicate quite definitely the existence of positive and not negative solitons. This is obviously due to the fact that a clearly expressed soliton (as is seen from experiments) has a large amplitude which exceeds severalfold the average thickness of the film. This is only for positive solitons (in negative solitons the amplitude, of course, cannot exceed the average thickness of the film, otherwise adhesion to the wall occurs; negative solitons, being of low amplitude, cannot be seen on a background of large-amplitude positive solitons).

2. The above-mentioned physical mechanism by which energy is pumped at low frequencies and dissipated at high frequencies, in principle, enables the energy to remain constant, which should make stationary motion of a soliton possible, i.e., a solution of Eq. (1.1) for which $\varphi(x, t) = \varphi(x - ct)$. For stationary motion (1.1) takes the form

$$-c\varphi' + 3\varphi' + \varphi\varphi' + \mathrm{Re}\,\varphi'' + W\varphi^{IV} = 0,\ (\ldots)' = d/du,\quad u = x - ct. \tag{2.1}$$

Integrating this equation with the boundary condition $\varphi \to 0$ as $u \to -\infty$ (or $u \to +\infty$), we obtain

$$(3 - c)\varphi + \varphi^2/2 + \mathrm{Re}\,\varphi' + W\varphi''' = 0,\ \varphi \to 0 \text{ for } u \to \pm\infty. \tag{2.2}$$

Hence we have two relations for a stationary soliton

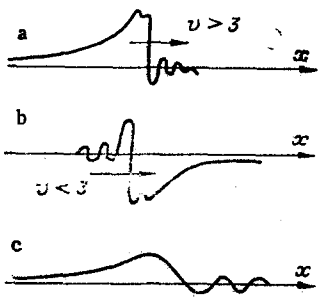

Fig. 1

$$(3-c)\int_{-\infty}^{\infty}\varphi du + \frac{1}{2}\int_{-\infty}^{\infty}\varphi^2 du = 0, \quad \int_{-\infty}^{\infty}\left(3-c+\frac{\varphi}{2}\right)\varphi^2 du = 0. \tag{2.3}$$

It follows from the second of the relations (2.3) that the brackets $(3-c+\varphi/2)$ alternate, and since when $n \to \pm\infty$, $\varphi \to 0$ we have $\sup|\varphi| > 2|c-3|$. Introducing the amplitude of the soliton $A = \sup|\varphi - 0|$, we obtain

$$A > 2|c-3|. \tag{2.4}$$

It follows from the first of the relations (2.3) that when $|c-3| \to 0, \varphi \to 0$. We therefore introduce the new variables $\psi = \varphi/2(c-3)$ and $\tau = u\,\mathrm{Re}^{1/2}W^{-1/2}$. We rewrite (2.2) in the new variables

$$\dddot{\psi} + \dot{\psi} - \mu\psi(1-\psi) = 0,\ \psi \to 0 \text{ as } \tau \to \pm\infty,\ (\dot{\ }) = d/d\tau, \quad \mu = (c-3)W^{1/2}\mathrm{Re}^{-3/2}. \tag{2.5}$$

We put forward some considerations in favor of the fact that as $\mu \to 0$, Eq. (2.5) has a monotonic solution which is not a soliton. In fact, as can be seen from (2.5), the function ψ_0 which satisfies the equation $\dot{\psi}_0 = \mu\psi_0(1-\psi_0)$, will also satisfy Eq. (2.5) with an accuracy to within terms $\sim\mu^3$ as $\mu \to 0$, which suggests that as $\mu \to 0$ the solution of (2.5) changes asymptotically into the solution of the equation $\dot{\psi} = \mu\psi(1-\psi)$, which is a monotonic solution of a nonsoliton type. Indeed, this indicates that the solution of problem (2.5) as $\mu \to 0$ can be sought in the form of an asymptotic expansion in powers of μ: $\psi = \psi_0 + \mu\psi_1 + \mu^2\psi_2 + \ldots$. To construct this expansion it is convenient to change to the coordinates of the phase plane ψ and $z = d\psi/ds$, where $s = \mu\tau$. In these coordinates problem (2.5), the initial approximation, and the asymptotic expansion take the following form:

$$z = \frac{\psi(1-\psi)}{1+\mu^2\dfrac{d^2}{d\psi^2}\left(\dfrac{z^2}{2}\right)}, \quad z(0) = 0, \quad z_0 = \psi(1-\psi), \ z = z_0 + \mu^2 z_2 + \mu^4 z_4 + \ldots. \tag{2.5'}$$

It is difficult to calculate the expansion coefficients z_n directly from (2.5'). Hence, assuming that this expansion reduces to a function having a finite derivative $(z^2/2)d^2/d\psi^2$ as $\mu \to 0$, and noting that in this case the right side of Eq. (2.5') is an analytic function of μ^2 and, of course, can be expanded in a converging series in powers of μ^2, we have $z = \psi(1-\psi)\left[1-\mu^2\dfrac{d^2}{d\psi^2}\left(\dfrac{z^2}{2}\right)\right] + o(\mu^4)$ or, after squaring,

$$z^2 = \psi^2(1-\psi)^2\left[1-\mu^2\frac{d^2}{d\psi^2}(z^2)\right] + o(\mu^4). \tag{2.5''}$$

Equation (2.5'') is linear with respect to the function $\Phi = z^2$ and has a solution which tends asymptotically (as $\mu \to 0$) to the function $\Phi_0 = \psi^2(1-\psi)^2$, i.e., has a finite derivative $d^2\Phi/d\psi^2$, which also ensures that (2.5') will change to (2.5''). We will not dwell on the solution of Eq. (2.5''), and will merely note that it can be constructed using the WKB method.

The above is confirmed by direct integration of Eq. (2.5') with the initial conditions $z(0) = 0$, $dz/d\psi|_{\psi=0} = \lambda/\mu$ $(\mu \ll 1, \lambda + \lambda^3 = \mu)$. It turns out that this monotonic solution ceases to exist when $\mu = \mu_0 \approx 0.2$. So, if a solution of (2.5) exists in the form of a soliton, we must have $\mu \equiv |c-3|W^{1/2}\mathrm{Re}^{-3/2} > \mu_0 \approx 0.2$. Using (2.4) we therefore obtain a lower limit of the values of the amplitude of a stationary soliton

$$A > 2\mu_0 W^{-1/2}\mathrm{Re}^{3/2} \tag{2.6}$$

(for a water film $W^{-1/2}\mathrm{Re}^{3/2} \approx 0.025\,\mathrm{Re}^{11/6}$, $A > (\mathrm{Re}/10)^{11/6}$). It can be seen from (2.6) that the amplitudes of solitons increase rapidly as the Reynolds number increases and become considerable ($A \sim 1$) when $\mathrm{Re} \sim W^{1/3}$ (for water when $\mathrm{Re} \approx 10$).

Following [6] we will now give an approximate solution of (2.5) for a stationary soliton (i.e., when $\mu > \mu_0 \approx 0.2$), using an analog of Galerkin's method. To be specific we will assume $\mu \sim (c-3) > 0$ (the case of negative μ – negative solitons – reduces, as was shown above, to the case of positive solitons with mirror-image coordinates).

It can be seen that (2.5) possesses asymptotics $\psi \to 0$ of the following form: $\psi \sim \exp \lambda\tau$, $\tau \to -\infty$ and $\psi \sim \exp(-\lambda\tau/2) \sin(\omega\tau + \delta)$, $\tau \to +\infty$, where λ is the positive root of the equation

$$\lambda^3 + \lambda = \mu, \quad \omega = \sqrt{1 + (3/4)\lambda^2}. \tag{2.7}$$

These asymptotics have the required ("soliton") form (see Fig. 1c, where we show a typical soliton observed experimentally). Hence, we wish to obtain a solution of Eq. (2.5) possessing such asymptotics. We form Galerkin's function from the asymptotics as follows:

$$\psi = A \exp \lambda\tau \cdot H(\tau) + B \exp\left(-\frac{\lambda\tau}{2}\right) \sin \omega\tau \cdot H(-\tau). \tag{2.8}$$

Here $H(\tau) = \begin{cases} 1, & \tau < 0, \\ 0, & \tau > 0 \end{cases}$ is Heaviside's function; because of the nonuniformity of the problem with respect to τ we can choose the origin of the reference system $\tau = 0$ in such a way that the initial phase (on the right) of the oscillating asymptotic is equal to zero (as was done).

The coefficients A and B (of which A plays the role of the amplitude of the soliton) can be found from the integral relations (2.3), which in the new variables have the form

$$\int_{-\infty}^{\infty} \psi d\tau = \int_{-\infty}^{\infty} \psi^2 d\tau = \int_{-\infty}^{\infty} \psi^3 d\tau. \tag{2.9}$$

Substituting (2.8) into (2.9) and introducing the notation $\alpha = \omega B/A$, we obtain the following algebraic system for α and A:

$$\frac{1}{\lambda} + \frac{\alpha}{1+\lambda^2} = \frac{A}{2\lambda}\left(1 + \frac{\alpha^2}{1+\lambda^2}\right), \quad 1 + \frac{\alpha^2}{1+\lambda^2} = \frac{2}{3}\lambda A\left(\frac{1}{\lambda} + \frac{2\alpha^3}{(1+\lambda^2)(1+3\lambda^2)}\right). \tag{2.10}$$

This system is solved for each λ. We thereby obtain $A(\lambda)$. To carry out the calculations it is convenient to introduce the new parameters $\lambda = \sinh \xi$, $\alpha = -\beta \cosh \xi$ instead of α and λ. System (2.10) then takes the form

$$1 - \beta \operatorname{th} \xi = \frac{A}{2}(1 + \beta^2), \quad 1 + \beta^2 = \frac{2}{3} A\left(1 - \frac{2\beta^3 \operatorname{th} \xi}{1 + 2\operatorname{th}^2 \xi}\right).$$

Omitting the elementary calculations we will give the result, according to which the quantity A is practically constant (independent of λ).

$$\text{As} \quad \mu \to 0 \ A \to 1.72, \ \text{as} \quad \mu = 0.8 \ A = 1.62, \ \text{as} \quad \mu > 1.5 \quad A \approx 1.56. \tag{2.11}$$

Since the amplitude of the soliton $A = A_\psi$ in the variables ψ is related to the dimensional amplitude $\tilde{A}$ by the equations $A = A_\psi = A_\varphi/2(c-3) = 6\tilde{A}/(\langle h \rangle 2(\tilde{c}/\tilde{u} - 3))$, where $\tilde{c}$ is the dimensional velocity of the soliton, and $\langle \tilde{u} \rangle$ is the velocity of the liquid in the film averaged over the cross section (serving as the unit of measurement of the velocities), we hence obtain

$$\tilde{c} = 3\langle \tilde{u} \rangle + \frac{3}{A} \frac{\langle \tilde{u} \rangle}{\langle h \rangle} \tilde{A}, \tag{2.12}$$

where $\langle \tilde{u} \rangle = \langle Q \rangle / \langle h \rangle = g\langle h^2 \rangle / 3\nu$.

As has already been stated, the value of A is practically constant. But it then follows from (2.12) that the velocity of the soliton depends linearly on the amplitude. This result agrees with the data obtained in [2, 3]. For a quantitative comparison of (2.12) with the experimental data these must be taken for the smallest possible value of Re (it was stated above that (1.1) holds for small Reynolds numbers). From [2, 3] we took data for the smallest of the Reynolds numbers given there (for Reynolds numbers corresponding to $\langle Q \rangle/\nu = 4$ and $\nu = 11.2 \cdot 10^{-6}$ m²/sec). We then have $\langle h \rangle = 5.4 \cdot 10^{-4}$ m and $\langle \tilde{u} \rangle / \langle h \rangle \approx 150$ sec^{-1}. Substituting this into (2.12) and then substituting A from (2.11) we obtain

$$\tilde{c} \approx 0.24 + 0.3(10^3 \tilde{A}) \tag{2.13}$$

($\tilde{A}$ is in meters and $\tilde{c}$ is in meters/sec). For comparison with experiment a curve of (2.13) is given in Fig. 2 together with experimental data taken from [2] (see graph A in Fig. 10d).

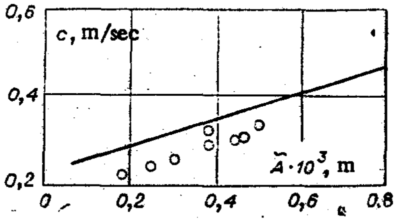

Fig. 2

Hence, we have the following results:

1) For each value of the average thickness of the film (or what is the same thing, the Reynolds number) there is a single-parametric family of stationary solitons, a typical form of which is shown in Fig. 1c. We can choose either μ (or λ) or the amplitude, or the velocity as the parameter of the family;

2) the amplitudes of the solitons of the family for each value of Re have a lower bound;

3) the velocity of a soliton is proportional to the amplitude;

4) the amplitudes of the solitons increase rapidly as the Reynolds number increases and when Re $\approx$ 10 (for water) they become comparable with the thickness of the film.

In conclusion it should be said that the soliton itself should be unstable in its plane (asymptotic, unperturbed) part in view of those mechanisms which make the plane Nusselt mode of the film unstable. In view of this instability perturbations should develop in the tail and forerunner of the soliton. Since in the main (central) part the motion has the form of a soliton, this indicates that a soliton, for those Reynolds numbers for which it is observed, is competitive and suppresses quasiharmonic waves, and it is natural to expect that perturbations of the asymptotes of a soliton will be converted in turn into a soliton etc., which leads in the final analysis to the formation of an irregular system of solitons. Precisely this pattern has been observed in [1-3] when the solitons are not excited artificially. Nevertheless, in a number of experiments [1-3] regular sets of solitons were produced artificially. Assuming that these can move in a stationary manner (with velocity c), we have Eq. (2.1) for them. Integrating (2.1) once we obtain

$$(3 - c)\varphi + \varphi^2/2 + \mathrm{Re}\,\varphi' + W\varphi''' = \mathrm{const}. \tag{2.14}$$

Multiplying (2.14) by φ and averaging over x (the quantity $\varphi\varphi''$ in this case disappears since it is a total derivative), we obtain $(3 - c)\langle\varphi^2\rangle + (1/2)\langle\varphi^3\rangle = \mathrm{const}\langle\varphi\rangle = 0$ (we have taken into account the fact that $\langle\varphi\rangle = \langle h/\langle h\rangle - 1\rangle \equiv 0$). Hence we have

$$c = 3 + \frac{1}{2}\frac{\langle\varphi^3\rangle}{\langle\varphi^2\rangle} = 3\left(1 + \frac{\langle\tilde{\varphi}^3\rangle}{\langle\tilde{\varphi}^2\rangle}\right). \tag{2.15}$$

Like all the results obtained above, Eq. (2.15) holds for small Reynolds numbers.

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